Lecture No -26

Example

Find the Fourier series for the function defined by

$f(x) = -x \qquad -\pi < x < 0$

$f(x) = 0 \qquad 0 < x < \pi$

$f(x) = f(x + 2\pi)$

The general expressions for a_0, a_n, b_n are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,dx = \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} = \frac{\pi}{2}$$

(b) To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\,\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx\right\} = -\frac{1}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{1}{n}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\left[\cos nx\right]_{-\pi}^{0} = -\frac{1}{\pi n^2}[\cos 0 - \cos n\pi]$$

$$= -\frac{1}{\pi n^2}\{1 - \cos n\pi\}$$

But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$

(n odd) $a_n = -\frac{2}{\pi n^2}$ (n odd) and $a_n = 0$ (n even)

(c) Now to find b_n

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\sin nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x \sin nx\,dx$

π −π

π −π 0

π −π

$\left(\frac{-\cos nx}{n}\right)$ 0 $\frac{1}{n}$ $\frac{1}{n}\left[\frac{\pi\cos n\pi}{n}\right]$

$\frac{1}{n}\left[\frac{\sin nx}{n}\right]$

0 $\frac{\cos n\pi}{n}$

= − π

$\{[x] + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\} = -\frac{1}{\pi}$

n −π −π

−π

$b_n = \frac{1}{n}$

So we have

π

(n even) and $b_n = -\frac{1}{n}$

2

(n odd)

$a_0 = \frac{\pi}{2}$; $a_n = 0$ (n even) and $a_n = -\frac{2}{\pi n^2}$ (n odd)

$b_n = -\frac{1}{n}$ (n even) and $b_n = \frac{1}{n}$ (n odd)

$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1} \ldots \frac{1}{\ldots} \right) \ldots \left(\frac{1}{\ldots} \; \frac{1}{\ldots} \; \frac{1}{\ldots} \right)$

n

$4\ \pi \Big(\cos x + {}_9 \cos 3x + {}_{25} \cos 5x + \ldots \Big) + \Big(\sin x - {}_2 \sin 2x + {}_3 \sin 3x - {}_4 \sin 4x + \ldots \Big)$

It is just a case of substituting n = 1, 2, 3, etc.

In this particular example, we have a constant term and both sine and cosine te

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f(x) sin nx dx

$\pi \ {}_{-\pi}$

$\pi \ {}_{-\pi}$

$\pi \ {}_{-\pi}$

$1 \ \pi \ 1 \ x^2 \ 0 \ 1 \ \pi \ 1$

$a_0 = {}_{\pi}\int f(x)\,dx = {}_{\pi}\int (-x)\,dx + {}_{\pi}\int 0\,dx = {}_{\pi}\int (-x)\,dx = {}_{\pi}\Big[-{}_2 \Big] = {}_2$

${}_{-\pi}$

(b) To find a_n

π

$-\pi \quad 0 \quad -\pi$

$\lfloor \ {}^0\ \rfloor \ {}_{-\pi}$

π

$a_n = \frac{1}{\pi}\int$

$f(x)\cos nx\,dx = \frac{1}{\pi}\int (-x)\cos nx\,dx + \frac{1}{\pi}\int 0\,dx = -\frac{1}{\pi}$

$\int x \cos nx\,dx$

$\pi \ {}_{-\pi}$

$\pi \ {}_{-\pi}$

0

$\pi \ {}_0 \ \pi \ {}_{-\pi}$

$\underline{\sin nx}\ {}^0 \quad 1 \quad 1$

$1\Big[\ \underline{-\cos nx}\ \Big]^0$

$= -\frac{1}{\pi}\Big\{ \Big[x \ \ {}_n \ \ \Big] - {}_n\int \sin nx\,dx \Big\} = -\frac{1}{\pi}$

$\Big\{ (0-0) - \frac{1}{n} \Big\}$

${}_{-\pi} \qquad {}_{-\pi}$

${}_{-\pi}$

$= -\frac{1}{\pi}\Big\{ \frac{1}{n}\Big[\ \underline{\cos nx}\ \Big]^0 \Big\} \qquad \underline{1} \Big[\ \underline{\cos nx}\ \Big]^0 \qquad \underline{1}$

$\Big\} = -\frac{1}{\pi n^2}$

$= -\frac{1}{\pi n^2}[\cos 0 - \cos n\pi]$

$\pi n = -\frac{1}{2}\{1 - \cos n\pi\}$

πn But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$

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$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$

$\left(\frac{-\cos nx}{n}\right)$ 0 $\frac{1}{n}$ $\frac{1}{\pi\cos n\pi}$

$\frac{1}{n}\left[\frac{\sin nx}{n}\right]$

$\frac{\cos n\pi}{\pi}$

$= -$

$\{[x] + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\} = -\frac{}{\pi}\{$

$n\,b_n = \frac{1}{\pi}$

So we have

(n even) and $b_n = -\frac{1}{}$

$\frac{}{2}$

(n odd)

$a_0 = \frac{}{2}$; $a_n = 0$ (n even) and $a_n = -\frac{}{\pi n^2}$ (n odd)

$b_n = -\frac{1}{}$ (n even) and $b_n = \frac{1}{}$

(n odd)

$f(x) = \frac{\pi}{} - \frac{2}{}\left(\mathbf{n} \quad \frac{1}{} \quad \frac{1}{} \right.$

n

$\left. \right) \quad \left(\quad \frac{1}{} \quad \frac{1}{} \quad \frac{1}{} \quad \right)$

$4 \; \pi \left(\cos x + {}_9 \cos 3x + {}_{25} \cos 5x + \ldots \right) + \left(\sin x - {}_2 \sin 2x + {}_3 \sin 3x - {}_4 \sin 4x + \ldots \right)$

It is just a case of substituting n = 1, 2, 3, etc.
In this particular example, we have a constant term and both sine and cosine terms.